Math 222A Lecture 16 Notes

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October 26, 2021

1 Fourier Inversion, Plancherel's Theorem, and Temperate Distributions

1.1 Fourier inversion

Last time, we introduced the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx.$$

We had an "inverse"

$$\mathcal{F}^{-1}v(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} v(\xi)\, d\xi.$$

Both $\mathcal{F}$ and $\mathcal{F}'$ are functions from $\mathcal{S} \to \mathcal{S}$, where $\mathcal{S} = \{\varphi : |x^\alpha \partial^\beta \varphi| \leq c_{\alpha,\beta}\}$ is the Schwartz space.

Theorem 1.1. *$\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id}$ on $\mathcal{S}$.*

Proof. Let's first try a brute-force approach and see what happens.

$$\begin{aligned}
\mathcal{F}^{-1}\mathcal{F}u &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi)\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \int_{\mathbb{R}^n} e^{-i\xi\cdot y} u(y)\, dy\, d\xi \\
&\stackrel{?}{=} \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi}\, d\xi\, dy
\end{aligned}$$

We know $\widehat{u}$ has rapid decay, so the first integral is well-defined. But it is not clear how we can integrate here. The $d\xi$ integral should evaluate to be $\delta_{x=y}$ in some way. Here is what we actually do:

$$= \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{\varepsilon}{2}\xi^2} \int_{\mathbb{R}^n} u(y)\, dy$$

Now we can legitimately apply Fubini's theorem.

$$\begin{aligned}
&= \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^n} \iint u(y) e^{i(x-y)\cdot\xi} e^{-\frac{\varepsilon}{2}\xi^2} \, d\xi \, dy \\
&= \lim_{\varepsilon \to 0} \int u(y) e^{-\frac{(x-y)^2}{2\varepsilon}} \varepsilon^{-n/2} \, dy \\
&= \lim_{\varepsilon \to 0} \int u * \varphi_\varepsilon \\
&= u,
\end{aligned}$$

where

$$\varphi_\varepsilon(y) = \frac{1}{(2\pi)^n} e^{-\frac{y^2}{2\varepsilon}} \frac{1}{\varepsilon^{n/2}} \xrightarrow{\varepsilon \to 0} \delta_0.$$

□

1.2 Isometry properties of $\mathcal{F}$ on L^2

Now let's shift our attention to L^2, with inner product $\langle u, v \rangle = \int u\overline{v} \, dx$.

Proposition 1.1. *The Fourier transform is unitary on L^2. That is,*

$$\mathcal{F}^* = \mathcal{F}^{-1}, \qquad (\mathcal{F}^{-1})^* = \mathcal{F}.$$

Proof.

$$\begin{aligned}
\langle \mathcal{F}, uv \rangle &= \iint e^{-ix\xi} u(x) \, dx \, \overline{v}(\xi) \, d\xi \\
&= \iint e^{-ix\cdot\xi} \overline{v}(\xi) \, d\xi \, u(x) \, dx \\
&= \iint \overline{e^{ix\xi} v(\xi)} \, d\xi \, u(x) \, dx \\
&= \langle u, \mathcal{F}^{-1} v \rangle.
\end{aligned}$$

□

This has the following consequence:

Theorem 1.2. *$\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an L^2-isometry.*

Proof. If we set $u = v$, we get

$$\|u\|_{L^2}^2 = \int |u|^2 \, dx = \|\mathcal{F}u\|_{L^2}^2.$$

□

We can use this to extend $\mathcal{F}$ to $L^2(\mathbb{R}^n)$ by density. If $u \in L^2$, find $u_n \in \mathcal{S}$ such that $u_n \to u$ in L^2. Then u_n is Cuachy in L^2, so $\mathcal{F}u_n$ is Cauchy in L^2. So $\lim_{n\to\infty} \mathcal{F}u_n =: \mathcal{F}u$.

Remark 1.1. The Hahn-Banach theorem says that we can extend operators that are densely defined, but in general, there is no guarantee of uniqueness.

However, it is not immediately clear that we can do this approximation of elements of L^2 by elements in $\mathcal{S}$.

Proposition 1.2. *If $u \in L^2$, then there exist $u_n \in \mathcal{D}$ such that $u_n \to u$ in L^2.*

This says that $\mathcal{D}$ is dense in L^2.

Proof. Step 1: Approximate u by compactly supported functions $u = \lim_{n\to\infty} u_n := u\mathbb{1}_{\{|x|\leq n\}}$.

Step 2: Regularize $u = \lim_{\varepsilon\to 0} u * \varepsilon$. Here, $\varphi \in \mathcal{D}$ iwth $|int\varphi = 2$, and $\varphi_\varepsilon = \varepsilon^{-n}\varphi(x/\varepsilon)$, so $\varphi_\varepsilon \to \delta_0$ as $\varepsilon \to 0$. So $u * \varphi_\varepsilon \to u$ in $\mathcal{D}'$ if $u \in \mathcal{D}'$ and in L^2 if $u \in L^2$. $\square$

So we get the following theorem:

Theorem 1.3 (Plancherel)**.** *$\mathcal{F} : L^2 \to L^2$ is an isometry.*

1.3 Temperate distributions

Can we extend $\mathcal{F}$ to any larger spaces? First, we will talk about the Fourier transform as a map $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$.

Definition 1.1. $\mathcal{S}'$, the space of **temperate distributions**, is the space of distributions which exend to continuous linear functionals on $\mathcal{S}$.

$u \in \mathcal{S}'$ if there is a constant c such that for $R\varphi \in \mathcal{S}$,

$$|u(\varphi)| \leq c \sum_{\text{finite}} p_{\alpha,\beta}(\varphi), \qquad p_{\alpha,\beta}(\varphi) = \sup |x^\alpha \partial^\beta \varphi|.$$

.

Heer is how we extend $\mathcal{F}$ and $\mathcal{F}'$ to $\mathcal{S}'$: For $u, v \in \mathcal{S}$,

$$\langle \mathcal{F}u, v\rangle = \langle u, \mathcal{F}^{-1}v\rangle,$$

so we have $\mathcal{F}u(\overline{v}) = u(\overline{\mathcal{F}^{-1}}v)$. Replacing v by $\overline{v}$ give $\mathcal{F}u(v) = u(\mathcal{F}v)$, where $u \in \mathcal{S}'$ and $\mathcal{F}v \in \mathcal{S}$. So we can define

$$\mathcal{F}u = u(\mathcal{F}v)$$

for $u \in \mathcal{S}', v \in \mathcal{S}$.

$\mathcal{S} \subseteq \mathcal{E}$, so $\mathcal{E}' \subseteq \mathcal{S}'$. If $u \in \mathcal{E}'$ (is compactly supported), then

$$\mathcal{F}u(\xi) = u\left(\frac{1}{(2\pi)^{n/2}} e^{-x\xi}\right).$$

So we see that $\mathcal{F} : \mathcal{E}' \to \mathcal{E}$. The moral here is that "$\mathcal{F}$ interchanges decay and regularity."

1.4 Examples of temperate distributions

When is a function a temperate distribution? If $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$,

$$u(\varphi) := \int u(x)\varphi(x)\,dx,$$

where $\varpi(x)$ is rapidly decreasing. So if $|u(x)| \leq c(1+|x|^N)$, then the integral is convergent.

Example 1.1. All rational functions are temperate distributions.

You should not get the idea that these are all the temperate distributions.

Example 1.2. Consider

$$u(x) = e^x \cos e^x.$$

Think of $u = \frac{\partial}{\partial x} \sin e^x = \partial_x f$. Then

$$u(\varphi) = -f(\partial_x \varphi),$$

where $\partial_x \varphi \in \mathcal{S}$ if $\varphi \in \mathcal{S}$. So a temperate distribution may not have much decay if it has enough oscillation, and there is a delicate balance between the two.

Here, if we have $x, \partial : \mathcal{S} \to \mathcal{S}$, we have extended $x, \partial : \mathcal{S}' \to \mathcal{S}'$.

1.5 The Fourier transforms of δ_0 and H

What is $\widehat{\delta}_0$?

$$\widehat{\delta}_0(\xi) = \delta_0\left(\frac{1}{(2\pi)^{n/2}} e^{ix\cdot\xi}\right) = \frac{1}{(2\pi)^{n/2}}.$$

Remark 1.2. People will often change the normalization constant in the Fourier transform to get $\widehat{\delta}_0 = 1$. So people will also replace $e^{ix\cdot\xi}$ with $e^{2\pi i x\cdot\xi}$. This is useful if you want to deal with Fourier series or if you want to make a distinction between the $\mathbb{R}^n$ of the input and the $\mathbb{R}^n$ of the ourput. These are actually the same space because $\mathbb{R}^n$ is the cotangent space fo $\mathbb{R}^n$. For more general spaces, the Fourier transform will not have the same input and output domain. We will not need to worry about this for our PDEs.

In 1 dimension, we have $\partial_x H = \delta_0$. Then

$$\mathcal{F}(\partial_x H) = \mathcal{F}(\delta_0),$$

which tells us that $-i\xi\mathcal{F}(H) = \frac{1}{(2\pi)^{n/2}}$. So we get that

$$\widehat{H} = \frac{i}{(2\pi)^{n/2}} \cdot \frac{1}{\xi}.$$

Take u compactly ussported in $[0, \infty)$. Then

$$\widehat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx.$$

Switch to complex numbers $\xi + i\zeta$. This integral ecomes

$$\int e^{-ix\xi + x\zeta} u(x)\, dx.$$

If $\zeta < 0$, we have exponential decay for $x > 0$. So $\widehat{u}(\xi)$ extends to a holomorphic function in $\{\operatorname{Im} z \leq 0\}$.

In this picture, we can think of

$$\widehat{H} = \frac{i}{(2\pi)^{n/2}} \cdot \frac{1}{\xi - i0}.$$

We can also look at

$$\widehat{H - 1} = \frac{i}{(2\pi)^{n/2}} \cdot \frac{1}{\xi + i0}.$$

So if we take the average, we get

$$\widehat{H - \frac{1}{2}} = \frac{i}{(2\pi)^{n/2}} \operatorname{PV} \frac{1}{\xi}.$$